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ON SOME THEOREMS IN THE NONLOCAL THEORY OF MICROPOLAR ELASTICITY

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Abstract—This paper is concerned with the linear nonlocal theory of micropolar elasticity. We have established a work and energy theorem and a uniqueness theorem without making any definiteness assumptions about the elastic moduli. A reciprocity theorem is also established.

1. INTRODUCTION

The nonlocal theory of micropolar elasticity is of recent origin. It was developed by Eringen (1973, 1975, 1976). It differs from the classical theory in fundamental hypotheses. The reference to the physical background of nonlocal theories may be found in Rogula (1982). Recently, some theorems on the nonlocal theory of elasticity have been established by Iesan (1977) and Altan (1989). The aim of this paper is to study the linear nonlocal theory of micropolar elasticity. First, we establish a work and energy theorem and a lemma representing a counterpart of Brun's theorem on the classical theory as given by Gurtin (1972), and prove a uniqueness theorem without definiteness assumption on the elastic coefficients. Then, by following the method established recently by Iesan (1989), a reciprocity theorem is obtained without either the use of the Laplace transform or the incorporation of the initial conditions in the equations of motion.

2. PRELIMINARIES

For a homogeneous and isotropic nonlocal micropolar elastic solid with a configuration Ω, bounded by a closed surface $\partial\Omega$, the basic equations of linear nonlocal theory of micropolar elasticity developed by Eringen (1973, 1975) consist of the equations of motion:

$$\sigma_{ji,j}+F_i = \rho\ddot{u}_i, \tag{1}$$

$$m_{ji,j}+\varepsilon_{ijk}\sigma_{jk}+M_i = I_{ij}\ddot{\varphi}_j, \tag{2}$$

the constitutive equations:

$$\sigma_{ij} = \int_\Omega \left(\lambda(|x-x'|)e'_{kk}\delta_{ij}+\mu(|x-x'|)[e'_{ij}+e'_{ji}]+\kappa(|x-x'|)e'_{ji}\right)\mathrm{d}\Omega', \tag{3}$$

$$m_{ij} = \int_\Omega \left(\alpha(|x-x'|)\psi'_{kk}\delta_{ij}+\beta(|x-x'|)\psi'_{ij}+\gamma(|x-x'|)\psi'_{ji}\right)\mathrm{d}\Omega', \tag{4}$$

and the geometrical relations:

$$e_{ij} = u_{i,j}+\varepsilon_{ijk}\varphi_k, \quad \psi_{ij} = \varphi_{i,j}. \tag{5}$$

In the above equations σ_{ij} is the Cauchy stress tensor, u_i is the displacement vector, ρ is the mass density, e_{ij} is the microstrain tensor, λ, μ, κ, α, β and γ are nonlocal material constants, φ_i is the microrotation vector, I_{ij} is the symmetric microinertia tensor, m_{ij} is the surface couple stress tensor, ε_{ijk} and δ_{ij} are alternating symbols, and F_i and M_i are, respectively, the body force vector and body couple vector. The notation of a Cartesian tensor is employed and a superposed dot denotes a time derivative, a comma followed by an index i denotes the derivative with respect to x_i. Primed variables depend on x' and t, and $d\Omega' = dx'_1\,dx'_2\,dx'_3$.

The solution of a mixed initial-boundary value problem in the linear nonlocal theory of micropolar elasticity for homogeneous, anisotropic solids is a process $\{u_i, \varphi_i\}$ that satisfies the basic equations (1)–(5) with the initial conditions:

$$u_i = \dot{u}_i = 0, \quad \varphi_i = \dot{\varphi}_i = 0, \tag{6}$$

and the boundary conditions:

$$u_i = \bar{u}_i \text{ on } \partial\Omega_1 \times [0, \infty), \quad \sigma_{ji} n_j = \sigma_i \text{ on } \partial\Omega_1^c \times [0, \infty), \tag{7}$$

$$\varphi_i = \bar{\varphi}_i \text{ on } \partial\Omega_2 \times [0, \infty), \quad m_{ji} n_j = m_i \text{ on } \partial\Omega_2^c \times [0, \infty), \tag{8}$$

where n_j is the unit outward normal to $\partial\Omega$, $\bar{u}_i$, σ_i, and m_i are prescribed continuous functions in the domains of their definition and

$$\partial\Omega_i + \partial\Omega_i^c = \partial\Omega, \quad i = 1, 2. \tag{9}$$

3. WORK AND ENERGY THEOREM AND UNIQUENESS THEOREM

In this section, we will state and prove a work and energy theorem which will be used to prove the uniqueness theorem.

Let V and E be the functions on $[0, \infty)$ defined by

$$V(t) = \frac{1}{2}\int_\Omega [\rho \dot{u}_i \dot{u}_i + I_{ij} \dot{\varphi}_i \dot{\varphi}_j]\, d\Omega, \tag{10}$$

$$E(t) = \frac{1}{2}\int_\Omega \int_\Omega [\lambda e'_{kk} e_{ii} + (\mu+\kappa) e'_{ij} e_{ij} + \mu e'_{ji} e_{ij}]\, d\Omega'\, d\Omega + \frac{1}{2}\int_\Omega \int_\Omega [\alpha \psi'_{kk} \psi_{ii} + \beta \psi'_{ji} \psi_{ij} + \gamma \psi'_{ij} \psi_{ij}]\, d\Omega'\, d\Omega, \tag{11}$$

where for convenience, the dependence of $|x - x'|$ is suppressed for nonlocal material constants and primed variables depend on x'.

Theorem 3.1 (Work and energy)

Let $\{u_i, \varphi_i\}$ be a solution of the mixed initial-boundary value problem defined by eqns (1)–(9) corresponding to $\{F_i, M_i\}$, then the work and energy theorem states that

$$\dot{V}(t) + \dot{E}(t) = \int_{\partial\Omega} [\sigma_{ji} \dot{u}_i + m_{ji} \dot{\varphi}_i] n_j\, dS + \int_\Omega [F_i \dot{u}_i + M_i \dot{\varphi}_i]\, d\Omega, \tag{12}$$

where n_j is the outward normal to the $\partial\Omega$.

Proof. Multiplying both sides of eqn (1) by $\dot{u}_i$, and integrating over Ω, we find that

$$\int_{\Omega} \rho \ddot{u}_i \dot{u}_i \,\mathrm{d}\Omega = \int_{\Omega} \sigma_{ji,j} \dot{u}_i \,\mathrm{d}\Omega + \int_{\Omega} F_i \dot{u}_i \,\mathrm{d}\Omega$$

$$= \int_{\partial\Omega} \sigma_{ji} \dot{u}_i n_j \,\mathrm{d}S - \int_{\Omega} \sigma_{ji} \dot{u}_{i,j} \,\mathrm{d}\Omega + \int_{\Omega} F_i \dot{u}_i \,\mathrm{d}\Omega. \quad (13)$$

Multiplying both sides of eqn (2) by $\dot{\varphi}_k$, and integrating over Ω, we obtain

$$\int_{\Omega} I_{kj} \ddot{\varphi}_j \dot{\varphi}_k \,\mathrm{d}\Omega = \int_{\Omega} m_{jk,j} \dot{\varphi}_k \,\mathrm{d}\Omega + \int_{\Omega} \varepsilon_{kji} \sigma_{ji} \dot{\varphi}_k \,\mathrm{d}\Omega + \int_{\Omega} M_k \dot{\varphi}_k \,\mathrm{d}\Omega$$

$$= \int_{\partial\Omega} m_{jk} \dot{\varphi}_k n_j \,\mathrm{d}S - \int_{\Omega} m_{jk} \dot{\psi}_{kj} \,\mathrm{d}\Omega + \int_{\Omega} \varepsilon_{kji} \sigma_{ji} \dot{\varphi}_k \,\mathrm{d}\Omega + \int_{\Omega} M_k \dot{\varphi}_k \,\mathrm{d}\Omega. \quad (14)$$

Now from eqns (13) and (14), we find that

$$\int_{\Omega} \rho \ddot{u}_i \dot{u}_i \,\mathrm{d}\Omega + \int_{\Omega} I_{kj} \ddot{\varphi}_j \dot{\varphi}_k \,\mathrm{d}\Omega = \int_{\partial\Omega} \sigma_{ji} \dot{u}_i n_j \,\mathrm{d}S + \int_{\partial\Omega} m_{jk} \dot{\varphi}_k n_j \,\mathrm{d}S$$

$$- \int_{\Omega} [\dot{u}_{i,j} - \varepsilon_{kji} \dot{\varphi}_k] \sigma_{ji} \,\mathrm{d}\Omega - \int_{\Omega} m_{jk} \dot{\psi}_{kj} \,\mathrm{d}\Omega + \int_{\Omega} F_i \dot{u}_i \,\mathrm{d}\Omega + \int_{\Omega} M_k \dot{\varphi}_k \,\mathrm{d}\Omega. \quad (15)$$

Noting that $\varepsilon_{kji} = \varepsilon_{jik} = -\varepsilon_{ijk}$ and using the relation (5), the above equation (15) may be rewritten as

$$\int_{\Omega} [\rho \ddot{u}_i \dot{u}_i + I_{kj} \ddot{\varphi}_j \dot{\varphi}_k] \,\mathrm{d}\Omega = \int_{\partial\Omega} [\sigma_{ji} \dot{u}_i + m_{jk} \dot{\varphi}_k] n_j \,\mathrm{d}S + \int_{\Omega} [F_i \dot{u}_i + M_k \dot{\varphi}_k] \,\mathrm{d}\Omega$$

$$- \int_{\Omega} \dot{e}_{ij} \sigma_{ji} \,\mathrm{d}\Omega - \int_{\Omega} m_{jk} \dot{\psi}_{kj} \,\mathrm{d}\Omega. \quad (16)$$

Changing index k into i in eqn (16), we get

$$\int_{\Omega} [\rho \ddot{u}_i \dot{u}_i + I_{ij} \ddot{\varphi}_i \dot{\varphi}_j] \,\mathrm{d}\Omega + \int_{\Omega} \dot{e}_{ij} \sigma_{ji} \,\mathrm{d}\Omega + \int_{\Omega} \dot{\psi}_{ij} m_{ji} \,\mathrm{d}\Omega$$

$$= \int_{\partial\Omega} [\sigma_{ji} \dot{u}_i + m_{ji} \dot{\varphi}_i] n_j \,\mathrm{d}S + \int_{\Omega} [F_i \dot{u}_i + M_i \dot{\varphi}_i] \,\mathrm{d}\Omega. \quad (17)$$

Using the constitutive equations (3) and (4), we find that

$$\int_{\Omega} \dot{e}_{ij} \sigma_{ji} \,\mathrm{d}\Omega = \int_{\Omega} \int_{\Omega} \dot{e}_{ij} [\lambda e'_{kk} \delta_{ij} + (\mu + \kappa) e'_{ij} + \mu e'_{ji}] \,\mathrm{d}\Omega' \,\mathrm{d}\Omega$$

$$= \int_{\Omega} \int_{\Omega} [\lambda e'_{kk} \dot{e}_{ii} + (\mu + \kappa) e'_{ij} \dot{e}_{ij} + \mu e'_{ji} \dot{e}_{ij}] \,\mathrm{d}\Omega' \,\mathrm{d}\Omega, \quad (18)$$

and

$$\int_{\Omega} \dot{\psi}_{ij} m_{ji} \, d\Omega = \int_{\Omega} \int_{\Omega} \dot{\psi}_{ij} [\alpha \psi'_{kk} \delta_{ij} + \beta \psi'_{ji} + \gamma \psi'_{ij}] \, d\Omega' \, d\Omega$$

$$= \int_{\Omega} \int_{\Omega} [\alpha \psi'_{kk} \dot{\psi}_{ii} + \beta \psi'_{ji} \dot{\psi}_{ij} + \gamma \psi'_{ij} \dot{\psi}_{ij}] \, d\Omega' \, d\Omega. \tag{19}$$

Substitution from eqns (18) and (19) into eqn (17) completes the proof.

Let us define

$$L(t_1, t_2) = \int_{\partial\Omega} [\sigma_{ji}(t_1) \dot{u}_i(t_2) + m_{ji}(t_1) \dot{\varphi}_i(t_2)] n_j \, dS + \int_{\Omega} [F_i(t_1) \dot{u}_i(t_2) + M_i(t_1) \dot{\varphi}_i(t_2)] \, d\Omega, \tag{20}$$

then to establish the uniqueness theorem, we need the following lemma:

Lemma 3.1.

For any $0 \leqslant t < \infty$, *we have*

$$V(t) - E(t) = \frac{1}{2} \int_0^t \big(L(s, 2t-s) - L(2t-s, s) \big) \, ds. \tag{21}$$

Proof. Using the divergence theorem, eqn (20) may be written as

$$L(t_1, t_2) = \int_{\Omega} \big(\sigma_{ji,j}(t_1) \dot{u}_i(t_2) + m_{ji,j}(t_1) \dot{\varphi}_i(t_2) \big) \, d\Omega + \int_{\Omega} \big(\sigma_{ji}(t_1) \dot{u}_{i,j}(t_2)$$

$$+ m_{ji}(t_1) \dot{\psi}_{ij}(t_2) \big) \, d\Omega + \int_{\Omega} \big(F_i(t_1) \dot{u}_i(t_2) + M_i(t_1) \dot{\varphi}_i(t_2) \big) \, d\Omega.$$

Taking into account eqns (1)–(4), the above equation becomes

$$L(t_1, t_2) = \int_{\Omega} \big(\rho \ddot{u}_i(t_1) \dot{u}_i(t_2) + I_{ij} \ddot{\varphi}_j(t_1) \dot{\varphi}_i(t_2) \big) \, d\Omega$$

$$+ \int_{\Omega} \int_{\Omega} \big(\lambda e'_{kk}(t_1) \dot{e}_{ii}(t_2) + (\mu + \kappa) e'_{ij}(t_1) \dot{e}_{ij}(t_2) + \mu e'_{ji}(t_1) \dot{e}_{ij}(t_2) \big) \, d\Omega' \, d\Omega$$

$$+ \int_{\Omega} \int_{\Omega} \big(\alpha \psi'_{kk}(t_1) \dot{\psi}_{ii}(t_2) + \beta \psi'_{ji}(t_1) \dot{\psi}_{ij}(t_2) + \gamma \psi'_{ij}(t_1) \dot{\psi}_{ij}(t_2) \big) \, d\Omega' \, d\Omega.$$

Taking into account the homogeneous initial conditions, we find that

$$\int_0^t \rho \ddot{u}_i(s) \dot{u}_i(2t-s) \, ds = \rho \dot{u}_i(t) \dot{u}_i(t) + \int_0^t \rho \dot{u}_i(s) \ddot{u}_i(2t-s) \, ds \tag{22}$$

$$\int_0^t I_{ij} \ddot{\varphi}_j(s) \dot{\varphi}_i(2t-s) \, ds = I_{ij} \dot{\varphi}_j(t) \dot{\varphi}_i(t) + \int_0^t I_{ij} \dot{\varphi}_j(s) \ddot{\varphi}(2t-s) \, ds \tag{23}$$

$$\int_0^t \int_{\Omega} \int_{\Omega} \big(\lambda e'_{kk}(s) \dot{e}_{ii}(2t-s) + (\mu + \kappa) e'_{ij}(s) \dot{e}_{ij}(2t-s) + \mu e'_{ji}(s) \dot{e}_{ij}(2t-s) \big) \, d\Omega' \, d\Omega \, ds$$

$$= -\int_\Omega\int_\Omega \big(\lambda e'_{kk}(t)e_{ii}(t)+(\mu+\kappa)e'_{ij}(t)e_{ij}(t)+\mu e'_{ji}(t)e_{ij}(t)\big)\,d\Omega'\,d\Omega$$

$$+\int_0^t\int_\Omega\int_\Omega \big(\lambda \dot{e}'_{kk}(s)e_{ii}(2t-s)+(\mu+\kappa)\dot{e}'_{ij}(s)e_{ij}(2t-s)+\mu\dot{e}'_{ji}(s)e_{ij}(2t-s)\big)\,d\Omega'\,d\Omega\,ds$$

$$= -\int_\Omega\int_\Omega \big(\lambda e'_{kk}(t)e_{ii}(t)+(\mu+\kappa)e'_{ij}(t)e_{ij}(t)+\mu e'_{ji}(t)e_{ij}(t)\big)\,d\Omega'\,d\Omega$$

$$+\int_0^t\int_\Omega\int_\Omega \big(\lambda \dot{e}_{kk}(s)e'_{ii}(2t-s)+(\mu+\kappa)\dot{e}_{ij}(s)e'_{ij}(2t-s)+\mu\dot{e}_{ji}(s)e'_{ij}(2t-s)\big)\,d\Omega\,d\Omega'\,ds \qquad (24)$$

and

$$\int_0^t\int_\Omega\int_\Omega \big(\alpha\psi'_{kk}(s)\dot{\psi}_{ii}(2t-s)+\beta\psi'_{ji}(s)\dot{\psi}_{ij}(2t-s)+\gamma\psi'_{ij}(s)\dot{\psi}_{ij}(2t-s)\big)\,d\Omega'\,d\Omega$$

$$= -\int_\Omega\int_\Omega \big(\alpha\psi'_{kk}(t)\dot{\psi}_{ii}(t)+\beta\psi'_{ji}(t)\dot{\psi}_{ij}(t)+\gamma\psi'_{ij}(t)\dot{\psi}_{ij}(t)\big)\,d\Omega'\,d\Omega$$

$$+\int_0^t\int_\Omega\int_\Omega \big(\alpha\dot{\psi}'_{kk}(s)\psi_{ii}(2t-s)+\beta\dot{\psi}'_{ji}(s)\psi_{ij}(2t-s)+\gamma\dot{\psi}'_{ij}(s)\psi_{ij}(2t-s)\big)\,d\Omega'\,d\Omega$$

$$= -\int_\Omega\int_\Omega \big(\alpha\psi'_{kk}(t)\dot{\psi}_{ii}(t)+\beta\psi'_{ji}(t)\dot{\psi}_{ij}(t)+\gamma\psi'_{ij}(t)\dot{\psi}_{ij}(t)\big)\,d\Omega'\,d\Omega$$

$$+\int_0^t\int_\Omega\int_\Omega \big(\alpha\dot{\psi}_{kk}(s)\psi'_{ii}(2t-s)+\beta\dot{\psi}_{ji}(s)\psi'_{ij}(2t-s)+\gamma\dot{\psi}_{ij}(s)\psi'_{ij}(2t-s)\big)\,d\Omega\,d\Omega'. \qquad (25)$$

Adding up eqns (22)–(25), we find that

$$\int_0^t L(s,2t-s)\,ds = 2V(t)-2E(t)+\int_0^t L(2t-s,s)\,ds,$$

which gives us the desired result.

Theorem 3.2. (Uniqueness)

Assume that ρ is strictly positive and I_{ij} is positive definite. Then the mixed initial-boundary value problem defined by eqns (1)–(9) *has at most one solution.*

Proof. It is sufficient to show that for $F_i = M_i = 0$ and homogeneous boundary and initial conditions, the solution is trivial. In fact, from the work and energy theorem and lemma 3.1, we have

$$V(t)+E(t) = 0,$$
$$V(t)-E(t) = 0.$$

Hence we find that $V(t) = 0$. With the hypotheses of the theorem, we find that $u_i = \varphi_i = 0$. This completes the proof.

4. RECIPROCITY THEOREM

Let $w(x,t)$ and $v(x,t)$ be functions defined on $\Omega\times[0,t_0)$. For $0<t<t_0$, the convolution of w and v is defined by

$$w * v = \int_0^t w(x, t-\tau)v(x, \tau)\,d\tau. \tag{26}$$

The notation for $i(x, t) = t$ and $v(x, t)$

$$i * v = \int_0^t (t-\tau)v(x, \tau)\,d\tau \tag{27}$$

will also be used in this section.

Let $R^r \equiv \{u_i^r, \varphi_i^r, \sigma_{ij}^r, m_{ij}^r, e_{ij}^r, \psi_{ij}^r\}$ be the solution corresponding to the external data system $Q^r \equiv \{F_i^r, M_i^r, \bar{u}_i^r, \bar{\varphi}_i^r, \sigma_i^r, m_1^r\}$, $r = 1, 2$; then we will prove the following lemma and the reciprocity theorem:

Lemma 4.1.
For $t_1, t_2 \in [0, \infty)$, $r, s \in \{1, 2\}$ *let*

$$\begin{aligned} A_{rs}(t_1, t_2) = &\int_{\partial\Omega} (\sigma_i^r(t_1)u_i^s(t_2) + m_i^r(t_1)\varphi_i^s(t_2))\,dS \\ &- \int_\Omega (\rho\ddot{u}_i^r(t_1)u_i^s(t_2) + I_{ij}\ddot{\varphi}_j^r(t_1)\varphi_i^s(t_2))\,d\Omega \\ &+ \int_\Omega (F_i^r(t_1)u_i^s(t_2) + M_i^r(t_1)\varphi_i^s(t_2))\,d\Omega, \end{aligned} \tag{28}$$

then

$$A_{rs}(t_1, t_2) = A_{sr}(t_2, t_1). \tag{29}$$

Proof. Using the divergence theorem, we find that eqn (28) may be written as

$$\begin{aligned} A_{rs}(t_1, t_2) = &\int_\Omega (\sigma_{ji,j}^r(t_1) + F_i^r(t_1) - \rho\ddot{u}_i^r(t_1))u_i^s(t_2)\,d\Omega \\ &+ \int_\Omega \sigma_{ji}^r(t_1)u_{i,j}^s(t_2)\,d\Omega + \int_\Omega m_{ji}^r(t_1)\psi_{ij}^s(t_2)\,d\Omega \\ &+ \int_\Omega (m_{ji,j}^r(t_1) + M_i^r(t_1) - I_{ij}\ddot{\varphi}_j^r(t_1))\varphi_i^s(t_2)\,d\Omega. \end{aligned}$$

Using eqns (1) and (2) in the above result, we find that

$$\begin{aligned} A_{rs}(t_1, t_2) &= \int_\Omega \sigma_{ji}^r(t_1)u_{i,j}^s(t_2)\,d\Omega - \int_\Omega \varepsilon_{ijk}\sigma_{jk}^r(t_1)\varphi_i^s(t_2)\,d\Omega + \int_\Omega m_{ji}^r(t_1)\psi_{ij}^s(t_2)\,d\Omega \\ &= \int_\Omega \sigma_{ji}^r(t_1)[u_{i,j}^s(t_2) - \varepsilon_{kji}\varphi_k^s(t_2)]\,d\Omega + \int_\Omega m_{ji}^r(t_1)\psi_{ij}^s(t_2)\,d\Omega \\ &= \int_\Omega (\sigma_{ji}^r(t_1)e_{ij}^s(t_2) + m_{ji}^r(t_1)\psi_{ij}^s(t_2))\,d\Psi. \end{aligned} \tag{30}$$

Using eqns (3) and (4) in eqn (30), we find that

$$A_{rs}(t_1,t_2) = \int_\Omega\int_\Omega \big(\lambda e'^{r}_{kk}(t_1)e^{s}_{ii}(t_2)+(\mu+\kappa)e'^{r}_{ij}(t_1)e^{s}_{ij}(t_2)+\mu e'^{r}_{ji}(t_1)e^{s}_{ij}(t_2)\big)\,d\Omega'\,d\Omega$$

$$+\int_\Omega\int_\Omega \big(\alpha\psi'^{r}_{kk}(t_1)\psi^{s}_{ii}(t_2)+\beta\psi'^{r}_{ji}(t_1)\psi^{s}_{ij}(t_2)+\gamma\psi'^{r}_{ij}(t_1)\psi^{s}_{ij}(t_2)\big)\,d\Omega'\,d\Omega$$

$$= \int_\Omega\left(\int_\Omega [\lambda e^{s}_{kk}(t_2)\delta_{ij}+(\mu+\kappa)e^{s}_{ij}(t_2)+\mu e^{s}_{ji}(t_2)]\,d\Omega\right)e'^{r}_{ij}(t_1)\,d\Omega'$$

$$+\int_\Omega\left(\int_\Omega [\alpha\psi^{s}_{kk}(t_2)\delta_{ij}+\beta\psi^{s}_{ji}(t_2)+\gamma\psi^{s}_{ij}(t_2)]\,d\Omega\right)\psi'^{r}_{ij}(t_1)\,d\Omega'$$

$$= \int_\Omega \big(\sigma'^{s}_{ji}(t_2)e'^{r}_{ij}(t_1)+m'^{s}_{ji}(t_2)\psi'^{r}_{ij}(t_1)\big)\,d\Omega'$$

$$= A_{sr}(t_2,t_1).$$

Theorem 4.1. (*Reciprocity*)

Let R^r be the solution corresponding to the external data system Q^r, $r = 1,2$, then

$$\int_{\partial\Omega} i * [\sigma_i^1 * u_i^2 + m_i^1 * \varphi_i^2]\,dS + \int_\Omega i * [F_i^1 * u_i^2 + M_i^1 * \varphi_i^2]\,d\Omega$$

$$= \int_{\partial\Omega} i * [\sigma_i^2 * u_i^1 + m_i^2 * \varphi_i^1]\,dS + \int_\Omega i * [F_i^2 * u_i^1 + M_i^2 * \varphi_i^1]\,d\Omega. \quad (31)$$

Proof. Taking $t_1 = t-\tau$, $t_2 = \tau$ in eqn (29) and integrating from 0 to t, we get

$$\int_{\partial\Omega} [\sigma_i^1 * u_i^2 + m_i^1 * \varphi_i^2]\,dS - \int_\Omega [\rho\ddot{u}_i^1 * u_i^2 + I_{ij}\ddot{\varphi}_j^1 * \varphi_i^2]\,d\Omega$$

$$+\int_\Omega [F_i^1 * u_i^2 + M_i^1 * \varphi_i^2]\,d\Omega$$

$$= \int_{\partial\Omega} [\sigma_i^2 * u_i^1 + m_i^2 * \varphi_i^1]\,dS - \int_\Omega [\rho\ddot{u}_i^2 * u_i^1 + I_{ij}\ddot{\varphi}_j^2 * \varphi_i^1]\,d\Omega$$

$$+\int_\Omega [F_i^2 * u_i^1 + M_i^2 * \varphi_i^1]\,d\Omega. \quad (32)$$

Taking the convolution of eqn (32) with i, we get the desired result with the help of the relations

$$i * \ddot{u}_i^r = u_i^r, \quad i * \ddot{\varphi}_j^r = \varphi_j^r,$$

and

$$w * v = v * w.$$

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